ONTOLOGICAL ASSUMPTIONS OF MATHEMATICS*

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Within the scope of contemporized standpoints in the philosophy of mathematics, which should be differentiated from the theory of the fundamentals of mathematics, we usually distinguish ontology, semantics, epistemology, methodology, and axiology. The problem of ontological assumptions of mathematics (in a broader sense: science, language) was contemporized in the 20th century by the position of R. Carnap and W. V. O. Quine, being in opposition to the so-called traditional essentialist model. In the further part of the present article, the discussion of these issues is based on the differentiation between the (predicates of) formal (conceptual) and real (material) existence. They can be described by a few opposed characteristics verbalized in existential sentences. The classical viewpoints of the ontology of mathematics (Platonism, nominalism, intuitionism, empiricism) are being contrasted with (ontological and epistemological) fictionalism, distinct from the traditional fictionalism represented by F. Nietzsche and H. Vaihinger.

1. Position of R. Carnap and W. V. O. Quine

The problem of ontological assumptions of mathematical theories (in a broader sense: scientific theories) have been recently contemporized by the discussions, which arose with regard to the attitudes of R. Carnap and W. V. O. Quine.[1] These two systems of ontology, being treated as a branch

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[1] Here, we mostly consider the following publications: R. Carnap, *Empiricism, Semantics and Ontology*, [in:] *Semantics and the Philosophy of Language*, Urbana 1952, pp. 208–228; *The Philosopher Replies*, [in:] *The Philosophy of Rudolf Carnap*, La Salle

of philosophy, were formed as an opposition to the so-called essentialism, which includes ontologies from before the period of revolution, after which philosophical systems were being treated metatheoretically.

From the point of view interesting for us, answers to the following questions are substantial: (1) What language is used for discussing ontological issues? (2) What is the difference between the ontological and non-ontological issues? (3) On what basis do we predicate the commitment of a theory towards some object or type of objects? (4) How do we interpret the ontological claims, and what is the basis of their significance? (5) What is the basis for determining the ontological questions?

In essentialism, in principle, the following answers to these questions were accepted: (1) Traditional ontological theories were formulated in everyday language complemented with technical terms, which were explained using the expressions from that language; (2) Ontological issues differed from non-ontological in the fact that they did not concern sufficiently general questions of existence, e.g. material objects, sensual data, or abstract beings. It is not a generality based on relation of inclusion, but on the so-called categories called the natural types of things. The objects are organized into such types. They are also identified on the basis of the common characteristics of objects belonging to this kind of orders (species, kinds) of things. The ontological issues are focused on the fact whether a satisfactory formulation of the world should distinguish some group of objects as being the most general in the meaning of the term presented here. The discussion concerning the answers to the questions (4, 5) shall indicate the divergence of attitudes with regard to the essence of the characteristics for individual categories. Nevertheless, the statement that the concept of ontology was — in traditional philosophy — based on the determination of categories of existing objects shall be right. (3) Presently, it is being pointed out (among others by Quine) that the main indicator of the ontological commitment in essentialism was constituted by names and referential expressions. It does not mean that such commitments existed for every object possessing a referential expression. To avoid commitments to unintended objects, even *ad hoc* interventions were used. In the traditional philosophy, the objective reference of a language was not clarified to such extent that a systematic criterion of its ontological assumptions could be determined. (4) The answers above were backed up quite commonly, whereas the remaining two had already been controversial. Nevertheless, their sum makes up for the

1963, p. 868 ff; W. V. O. Quine *Word and Object*, Cambridge, Mass. 1960; *Ontological Relativity and Other Essays*, New York 1968; *From the Point of View of Logic* (in Polish), Warszawa 1969, (translated from English by B. Stanosz).

essentialist standpoint. True ontological claims were interpreted in such a way that they revealed some attribute of reality. The categories determined by this reality were attributed a realistic interpretation. There also exists an objective evidence, not relativized to the context, situations or targets, which indicates true categories in the absolute, prelinguistic sense. Such categories are logically prior to the categories determined by the choice of certain conceptual or linguistic scheme. A set of categories is determined by reality, which either is or is not expressed by a language. If language expresses a prelinguistic set of categories, then isomorphy between it and the structure of reality takes place, otherwise such relation does not exist. In essentialism, ontological issues are settled on the basis of extralinguistic nature of things. (5) The answer to this question is determined by the content of the preceding answer. Settling the ontological issues refers to the arguments as convincing as their premises describe the nature of reality correctly. The system of premises is not based upon decisions but on discoveries and in case they are correct, the claims of opposed standpoints inconsistent with them shall be groundless. It is also important that such discoveries are made in an experimental or purely rational way. They are also independent of the language used, targets and human situation.

The standpoint determined by those answers was not generally approved. From the very beginning, the representatives of (extreme) nominalism, who did not approve upon the essentialist formulation of categories were in opposition to it. In the 20th century, Quine's and Carnap's proposals appeared to be alternative to essentialism.

(1Qu) According to Quine, first-order predicate calculus (OPC) is equivalent for ontology. It is accurate enough to isolate and discuss ontological issues. It appears that significant controversies concerning the nature of ontology can be expressed by selecting a language best with regard to better explanation of the structure of ontological issues. (2Qu) Quine does not respect the difference between the ontological and scientific issues. He does not accept the differentiation between the science and ontology to be substantial. Philosophy undertakes the issues, which are, or should be, solved in the field of science. Abolishing the borders between philosophy and science constitutes one of the aspects of his claim concerning the essential unity of knowledge. (3Qu) Quine's criterion for ontological commitment (theory) in a simplified form subjects existence to the values of its bound variables; existence would be expressed by existential quantification. To be a little more accurate, we can say that ontology, towards which the (interpreted) theory is committed, contains only the set of such objects, through which its bound variables, constructed on these objects, are running in such a way that its claims are true. (4Qu) Ontological sentences are assumptions.

Theories are not more accurately determined by attainable data. Like the ontological statements, ontologies are relativized towards theory. It is not an attempt towards reduction of ontological theories, as Quine is convinced that both scientific and ontological theories are based on similar evidence. (5Qu) Besides the ontological criterion for commitment, Quine is putting forward the criterion of choice from among the ontological standpoints. If the first is applied to theory, establishing the relation between theory and objects, the second allows to estimate theories on the basis of simplicity bound to certain elements of conservatism being expressed in attempts to fit new data to hitherto existing theories. The result of such comparison does not lead to single result due to ambiguity of "simplicity".

(1C) Carnap agreed with Quine with regard to OPC. In fact, however, he thought that the language of ontology is constituted by the predicate calculus, but not only of the first order (MPC). (2C) Carnap distinctly differentiates ontology from science. Ontological issues concern the questions of existence of the objects classified into categories. (3C) Carnap accepts Quine's criterion of ontic commitment. (4C) Ontological claims are interpreted as a certain case of Carnapian external (non-cognitive) questions, metatheoretical in relation to the language of the theories opposed to the internal, theoretical questions. In ontology, philosophical questions of the existence of objects, their types, are being considered. (5C) Carnap, like Quine, is treating the ontological claims as assumptions. Objects, towards which a theory is committed, are relativized to a notional scheme, using which the theory is being established. Neither this scheme nor the theory are correct in an absolute sense. The objects concerned are not discovered, but supposed, they are being postulated and at the same time real, from the view-point of the building process of the theory.

While comparing these attitudes, one should note that Carnap and Quine agree to the fact that referential expressions of colloquial language are not adequate indicator of ontological assumptions of a theory. Contrary to the traditional realism, Carnap and Quine represent conventionalism in the way they treat categories. They reject realistic interpretation of ontological statements, which are pragmatically determined assumptions for them. Ontological questions are also pragmatically settled using evidences and linguistic means. The difference between Carnap and Quine lies mostly in the status of ontological and scientific statements. As Quine — contrary to Carnap — does not approve of dichotomy of analytic and synthetic statements, all the claims — including ontological and scientific — are hypothetical.[2]

[2] In connection with the separation of ontology from non-ontological disciplines J. Cornman (*Metaphysics, Reference and Language*, New Haven 1966, XVIII–XIX) no-

2. Mathematics versus existence and reality

Let us use the differentiation between the formal existence called also conceptual or ideal existence from the factual existence also called concrete or material. It is based on the differentiation of constructs, which are e.g. concepts or statements, from factual objects, which are concrete objects or their states. Then, we can say that e.g. numbers belong to the set of constructs and exist formally, whereas electrons belong to the set of material objects and exist factually. Then, we have both a predicate of formal (conceptual) existence and a predicate of real (material) existence. These predicates seem more appropriate in the discussion of ontological issues than the "existential" quantifier usually used. When we use this appropriately specified differentiation, the expressions like "abstract objects do not exist" (H. Field) or "mathematics and science concern non-existing objects" (R. Routley) will seem to be more comprehensible.

Real existence is characterized by absoluteness, whereas formal existence is not susceptible to changes, it is relativized to context, it also is conceptually perceptible.

While real objects exist in an absolute way, every construct exists in one context and not in another, e.g. in effect of proving a theory. Natural numbers exist e.g. formally in the theory of numbers but they do not in the lattice theory, whereas the existence of electrons, cells, societies is not relativized to the context, they exist in an absolute way.[3]

Contrary to constructs, material objects are changeable. Contrary to e.g. numbers, photons move as long as they exist. The equations of the movement of photons are given and not of numbers. There also a reverse dependence takes place: variable objects exist really, whereas something immutable exists formally (conceptually). However, in the tradition originating in the eighteenth-century mathematics, derivative of a function was, informally, characterized as a frequency of variation of this function, but these are not the function that vary, these are the variable properties of concrete objects represented by them.

According to the representatives of intuitionism, mathematical objects do not exist independently, but they are the constructs of intuition. The

ticed certain specific nature of philosophy concerning ontological commitment of colloquial language and of the language used in philosophy. See also: B. G. Norton, *Linguistic Framework and Ontology*, The Hague 1977 pp. 17–24, 63, 65, 74–75; B. Stanosz, *Introduction* [in:] W. V. O. Quine, *Limits of Knowledge and Other Philosophical Essays* (in Polish), Warszawa 1986, pp. 7–14; W. V. O. Quine, *From the Point of View of Logic*, pp. 28–34.

[3] Although physical objects are existing in an absolute way, their certain properties and variations are relativized to the reference system.

latter, on the other hand, is a process, whose products are equally temporal as physical objects.[4] It is not right, when we want to notice the research, which is subjective, and its results, which may be universal. Mathematical objects — created or discovered — are atemporal creations of reason, whereas material objects are continuously variable and meet the factual laws, e.g. physical laws. Mathematical claims, more generally, truths of the reason, contrary to the truths of the facts are timeless. We differentiate e.g. mathematical statement: "π is a transitive number" from the factual statement: "in the previous century it was proved that π is a transitive number".

Indispensable and satisfactory condition for being a construct is its conceptual perceptibility. In realistic epistemology, which we adopt, this condition is indispensable for the real existence of objects. Physical world had been existing before intelligent beings appeared in it. On the other hand, there do not exist mathematical object prior to the appearance of beings capable of composing them. Let us assume that x is a construct only when there does really exist a being, who is capable of perception x as a conceptual system or its component.

Existence in mathematics is also addressed with the requirement of theoreticality.[5] It is determined within a consistent theory, in which there exist clear concepts. The constructs, which occur in inconsistent theories with blurred expressions, are deprived of mathematical existence.

Keeping in mind the characteristics of formal existence presented, Plato was right when he claimed that ideas are immaterial and invariable, but he was wrong claiming that they exist independently. On the other hand, Hegel was wrong both when he claimed that ideas exist independently and when he exposed their internal activity revealed in their transition through three phases of dialectics (thesis, anti-thesis, and synthesis).

Further explanation of formal and real existence shall be carried out together with the characteristics of existential statements. They are equally important in both formal and factual sciences. Detailed characteristics of objects carried out within factual sciences is preceded by ascertainment or clear supposition concerning their existence.[6] In mathematics, the exis-

[4] Therefore, they eliminate the concept of current infinity as unintelligible, e.g. M. Dummet.

[5] The requirement of the consistency of a mathematical theory and the possibility of developing it in a deductive way enforces limitations upon its objects determined by the system of theory axioms. See: D. Gierulanka, *The Problem of Specific Nature of Mathematical Cognition* (in Polish), Warszawa 1962, p. 185 ff.

[6] In ideology or in pseudo-science, there are objects described, whose existence has not been established or whose existence cannot be established.

tence of a certain type of objects is being postulated or proved before a statement met by them is put forward or proved. (At the starting point of constructionist mathematics, the objects concerned are being constructed using clearly determined procedures like e.g. calculation). In fact, the antecedent of a general mathematical statement is (silently or distinctly) an existential statement avoided only when it is adopted *a priori*. In the simplest case, the form of such statement is as follows: if x does exist, then if x has the property F then it also has the property G.

Existential statements state that certain object (objects) is contained in a certain set, class or category determined by a consistent theory. The well-known statement concerning the existence of an infinite number of prime numbers is expressed in the statement: a set of prime numbers constitutes an infinite subset of the set of natural numbers. General form of this thesis is also definitionally percepted. If x is a construct, then x exists mathematically iff for some C, C is such a set, class or category that x is contained in C determined by a precise and consistent theory.

In factual fields of knowledge, the problems of existence and existential statements are analyzed using different methods. The formula $(\exists x)Fx$ — there do exist F's — is attributed with different interpretations: ontological (some real objects possess the property represented by the predicate F), psychological (in a given time someone assumes that certain objects possess the characteristics F), pragmatic or intuitionist (it is possible to find or construct objects possessing the characteristics F). These interpretations go beyond the standard mathematical interpretation of this formula, according to which certain constructs possess the characteristics F, i.e.: set, class, category determined by F is not empty.[7]

In mathematics, the problem of existence is seen most sharp in reference to infinite sets. The farthest-reaching objections are lodged by intuitionists, whereas the representatives of formalism claim that one should carry on as if such sets existed really (A. Robinson). Following the attitude adopted here, both finite and infinite sets, and in fact all the constructs, exists in a formal way, providing they are well-determined and at the same time conceptually perceptible by a competent scholar in the form of a set, class or category. The last property of constructs (susceptibility to conception) presents difficulties even in case of "small" mathematical objects. They are being revoked, paying attention to the fact that mathematical objects are systemic and in compatible with specific formulae. Let us present the following expressions of number theory as an example: $n!$, $(n+1)! =$

[7] The interpretations of mathematical formulae given are justified only in extra-mathematical contexts, e.g. in physics, astronomy.

$= (n+1)n!$, $(n+1)!/n! = n+1$. In mathematics, no objections are being raised with regard to formal existence of $n!$ and the remaining equalities. They are objects well-determined by the formulae (definitions, theses) of the theory of numbers. Therefore, they meet the systemic condition and the condition of compatibility with specified formulae of a theory.[8]

Maintaining the differentiation between the conceptual and real existence, we have assumed that material objects are real and not mathematical constructs. Ontology of real objects is not in force in the field of constructs. Epistemology applicable to constructs representing material objects is realistic considering the latter, whereas considering the constructs of logic and mathematics it is fictionalist.

When the relationship between the formal and real existence is concerned, we maintain that the constructs are the products of thoughts, which contradicts Platonism. At the same time we do not identify constructs with mental processes. Another possible relationship between the formal and real existence is realized through the relation between a mathematical and physical (chemical, biological, social) system. In other words, asking about the way in which mathematics is related to the world constitutes a particular case of a more general question concerning the relationship between ideas and the external world. Contrary to objective idealism on one hand, and to empiricism and pragmatism on the other, we maintain that mathematics is not ontologically committed and therefore it may constitute a tool for constructing theories representing various kinds of objects.[9] As a matter of fact, the same conceptual systems of mathematics, also called "structures", every time equipped with different interpretations concern different fields of research. Such interpretations based on the semantic assumptions do not belong to pure mathematics — they are a fragment of factual theories. The objective reference of the statements of pure mathematics is made by conceptual objects, e.g. sets, functions.[10]

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[8] Owing to the reference of constructs to sensual experience or action, the representatives of empiricism and pragmatism oppose against infinite sets. In creating and operating the infinite constructs, however, we use finite formulae (e.g. iteration, recurrence) met by these constructs.

[9] The obligation of the laws of the algebra of sets, e.g. is not dependent on the nature of the elements of these sets. The statements determining the relation "smaller than" are also independent of the nature of the elements of the set, on which it is being determined.

[10] Pure mathematics investigates conceptual systems (W. E. Harnet, M. Bunge) also called structures (P. Bernays, N. Bourbaki), or the members of such systems with *a priori* (conceptual) means to establish the regularity or schemes met by these objects and justified only through proof. Applied mathematics undertakes the issues occurring in factual, polytechnic and humanistic sciences, analyzing them basing on the constructs of pure mathematics. If, however, in applied mathematics, such constructs are being put

If mathematics does not represent the world, and if it is not the most general science about this world either, then in particular it does not explain real changes. Every mathematical description of such change contains semantic assumptions, owing to which mathematical objects represent non-mathematical objects, e.g. their properties. The relation between the mathematical constructs and variable objects is identical with the relation between mathematical operations to processes. The question of contrasting invariable constructs with changing reality is still controversial (Plato, Bergson, and J. Lambek with reference to topology). All those, who question this opposition should be addressed with the fact that the root of every advanced field of knowledge is constituted by a system of equations, containing mathematical objects insusceptible to changes (e.g. Maxwell equations, laws of population genetics). The reason justifying the representation of a change by invariable mathematical objects is the fact that the objects concerned occur in factual sciences together with semantic assumptions, e.g. correspondence rules. They determine the referents of the constructs and their properties. Pure mathematics does not offer such assumptions, as it is not ontologically committed. They are discovered by semantic analysis of factual theories. Although pure mathematics is ontologically noncommitted, it constitutes conceptual framework, which together with semantic assumptions are used for explaining the real world. As an *a priori* science, it is indispensable to build *a posteriori* science concerning the same world.[11]

As mathematics does not represent the external world, then it is not objective in a semantic or epistemological sense, but in methodological sense, which guarantees the systemic characteristics as well as the correctness of

forward, they are treated not like aims but like means. Applied mathematics differs from the pure one: (a) in the source of issues, which is either external (applied) or internal (pure) as compared with mathematics; (b) final referents, which are either real objects (applied) or constructs (pure); (c) task, which is either aiding the non-mathematical disciplines (applied) or enriching mathematics itself (pure). I. Niiniluoto (*Is Science Progressive?*, Dordrecht 1984, p. 207) indicates the differentiation between pure and applied geometry, which took place in the 19th century.

[11] Pure mathematics is not ontologically committed, it does not refer to reality. If not, it would be an *a priori* and universal science about the world. However, it constitutes the basic language of science and technology, and a store of concepts and proof means. Therefore, it is not a factual science, but an indispensable means for obtaining accurate and thorough factual knowledge. This conception of nature and the role of mathematics is called instrumental formalism or formal instrumentalism by M. Bunge. From epistemology, instrumentalism and formalism it differs in the fact that: (a) it does not establish that mathematical formulae are not statements but rules or instructions; (b) does not make practice the criterion for assessment; (c) does not reject those mathematical ideas, which have not been applied yet, nor those, which have already lost their application in science and technology.

its objects. From the epistemological or semantic viewpoint, mathematics is neither subjective (intuitionism) nor objective (Platonism, dialectic materialism), but neutral as this kind of construct status is determined neither by subjective experience nor by independently existing world. Existing formally, they are used for examining this world.

In the ontological plane, the objects of mathematics remain equal to the works of art, as they are fictitious creations. The difference between them lies not in the epistemological plane, as some mathematical constructs — like artistic fictions — are idealizations of real objects or their properties. Mathematics does not differ from art considering certainty, although the will to obtain definite certainty, motivated mathematical research, particularly in the field of the foundations of mathematics. However, subsequent critical situations in the latter, multitude of nonequivalent versions of set theory, continuous controversies around the axiom of choice indicate that definite certainty is unavailable. Certainly, mathematics is sure to a greater degree than factual sciences, which, on the other hand, possess greater degree of sureness than other fields of research. Moreover, the foundations of mathematics do not possess the status of invariability and uniqueness.

Fictional character of the objects of mathematics substantially differs from fictional objects of other type. (1) Although mathematical objects are deprived of factual reference, they are not random products of invention due to the limitations imposed on them by definitions and theses of formal sciences. (2) They exist formally on the strength of postulates or proofs and not on the strength of arbitrary decisions. (3) Mathematical objects are theoretically in opposition to e.g. literary fiction. (4) Theories and objects of formal sciences are fully rational and the theses of these sciences must be rationally justified. (5) We abandon hypotheses, on which mathematical theories are based if it appears that they lead to controversies or trivialities. (6) Mathematical theories make a coherent system, which is revealed e.g. by the use of algebraic methods in logic, or use of analysis in the theory of numbers. (7) From semantic or epistemological viewpoint, mathematics is neither objective nor subjective but neutral and ontologically noncommittal, although the process of creating mathematics by real creators is subjective. (8) Mathematical theories and objects are applied in science, technology and arts, being at the same time socially neutral.

The characteristic of existence in mathematics given above, as well as its reference to reality shall be supplemented by remarks concerning “existential” quantifier ($\exists$). It is being suggested (M. Bunge) that it is better to call this operator undetermined particularizer or quantifier as different from universalizer ($\forall$) and individualizer or descriptor (ι). Logicians, in most cases, undertake Quine’s suggestion that existence is expressed by existential quantifier. This operator does not report, however, the differentiation between formal and real existence introduced here after M. Bunge. Independent of specifying these two concepts of existence we do not read symbol $\exists$ as "exists". E.g. the formula "$(\exists x)$ x is a phantom" we do not read "phantoms exist" but "some individuals are phantoms, i.e. the creations of imagination".[12]

Explaining this question, let us pay attention to two statements. (1) There exist malicious phantoms. (2) Some of the phantoms are malicious. According to Russell, Quine and majority of logicians statements (1) and (2) should be formalized as follows: $(\exists x)(Gx\&Wx)$. According to the position adopted here, these two statements are different in this sense that the first — contrary to the second — is existential. Therefore, formalizing statement (1) the existence predicate ($E_{\mathbf{M}}$) should occur, which possesses individual variable x like other predicates. Thus we have the expression: $E_{\mathbf{M}}x\&Gx\&Wx$, where **M** symbolizes mythology. We understand this formula in such a way that in mythology, there exist individuals, which are malicious phantoms. Preceding this formula with a particularizer, we yield a statement that some of the phantoms existing in mythology are malicious $(\exists x)(E_{\mathbf{M}}x\&Gx\&Wx)$. Only in this statement, ontological commitment occurs, whereas statement (2) is ontologically neutral.[13]

The remarks above lead to a few interesting conclusions. (1) In formal expressions of existential statements there should occur a symbol denoting existence. (2) Repealing Quine's objection addressed to the logic of the second order considering its — apparent — "excessive ontological commitment" is also the effect of de-ontologization of the "existential" quantifier. Subsequent consequences are already of a more technical nature and are concerned with the law of subalternation of a logical square and of the axiom: $Fa \vdash (\exists x)Fx$. If we accept the reinterpretation of small quantifier presented here, then we shall modify the form of the record of these formulae. This reinterpretation is also concerning the nature of elementary logic, as — contrary to Quine — we shall say that it is not ontologically committed, but neutral. The semblance of such commitment originates from Quine's interpretation of this operator. Meanwhile, the existence in formal sciences is not determined by this operator, but it is the problem of postulating or proving.[14]

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[12] Alternative interpretation of the formula "$(\exists x)Gx$" says: the formula "Gx" is satisfiable, or equivalent: certain reductions of the formula "Gx" are true. These two, occurring in *Principia Mathematica*, interpretations are equivalent. Certain object has the property G, when certain reductions of the expression "Gx" are true and vice versa.

[13] If mythology is accepted to be true, engagement is serious, if not, it is only apparent.

[14] M. Bunge, *Treatise on Basic Philosophy*, Dordrecht 1985, VII, pp. 22-40,40-46,86.

3. Ontology of mathematics as a branch of the philosophy of mathematics

Here, let us drop the empirical issues undertaken within such factual disciplines as history, psychology, sociology of mathematics. Historically, many directions of the philosophy of mathematics have been formed, of which some are timely till present.[15] In the philosophy of mathematics, the issues of the ontological status of mathematical objects are being discussed (ontology), the issues of objective reference of mathematical theories and truth in mathematics (semantics), the question concerning the nature and sources of mathematical cognition (epistemology), and the problems concerning in particular proving in mathematics and its application (methodology).[16]

Platonism is included in the group of classical directions in the philosophy of mathematics, which is a kind of objective idealism, nominalism, intuitionism, which is a variant of subjective idealism, and empiricism, and in particular — pragmatism.[17]

Platonism, also called realism, is a philosophy of logistic strategies of fundamental research (among others G. Frege, K. Gödel). It also is a spontaneous philosophy of mathematicians in a way, as majority of them seem to share the conviction that they discover and investigate objects existing outside the space and time, which are independent of any mind. They would exist even if there were no cognition subject at all (N. Goodman). It is wrong from the point of view of a substantially different real (material) and formal existence. Research procedure of mathematicians, however, is as if constructs existed independently. Majority of mathematicians respect Platonism. E.g. the Bourbaki group, official representatives of formalism, seem to be convinced about the reality typical for mathematics. Platonism

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[15] In case of this issue see: e.g. H. Putnam, *Philosophy of Mathematics: A Report*, [in:] Current Research in Philosophy of Science, East Lansing 1979, pp. 386–398.

[16] In philosophy of mathematics, beside ontology and epistemology, we differentiate axiology, understood as the theory of values regulating the cognitive process in mathematics. See: A. Lubomirski, *On Generalization in Mathematics* (in Polish), Wrocław 1983, pp. 41–61.

[17] In the theory of the fundations of mathematics (constituted by mathematics and logic in the broader understanding) we differentiate three classical directions namely: logicism, formalism and intuitionism with constructivism. In effect of mutual interactions, nowadays it is difficult to speak about pure forms of these directions. In case of philosophical questions (e.g. the nature of mathematical objects, the method of learning them) they were associated with a specific philosophical standpoint: logicism with idealism (in particular with Platonism), formalism with nominalism, and mathematical intuitionism (together with constructivism) with philosophical intuitionism (e.g. Kant's) and even with operationism. None of these directions was completely in agreement with its philosophical sources. Very often these directions are being treated as research strategies independent of the historical philosophical issues. See: M. Bunge, *Treatise*, pp. 95, 97–98, 107.

for sure correctly explained unversality of mathematics, invariability and immateriality of its objects.

Nominalism, being in opposition to realism, is a formalistic philosophy (e.g. S. Leśniewski, L. Chwistek). Mathematical and physical objects are equally treated. There is no differentiation between conceptual and physical existence. General signs, as opposed to general ideas, which are not denoted by these signs. As mathematics operates with symbols, which are always conventional, its formulae are true by the force of linguistic conventions similar to game rules.

Nominalism had fascinated mostly those representatives of formalism, who were interested in more advanced branches of mathematics. D. Hilbert, representative advocate of formalism, did not accept the thesis of consequent nominalism saying that mathematical objects did not exist in fact, there only exist words, which mean nothing. Hilbert, on the other hand, noticed the requirement for abstract object, formal truth, which was expressed in his contribution to the theory of models. Together with the constitution of semantics, nominalist thesis concerning the exclusively "formal" syntactic nature of axiomatic system was questioned. Nominalism, understood strictly, postulates the elimination of uncountable sets (e.g. the set of real numbers together with geometrical interpretation), as most of the elements of such sets have no names. In the liberal form of nominalism, the existence of natural numbers (in Platonic or constructivist sense) is accepted, whereas the existence of sets is negated. This term ("set") is only a comfortable method of speaking about natural numbers and, in principle, it can be eliminated from the language of mathematics at the cost of complicated utterances.

When the nominalist thesis concerning the existence of individuals only is justified with regard to material objects, Platonic thesis concerning the reality of universals is not justified on the ground of factual sciences. Generalization of this onto all the objects would not be justified. Nevertheless, it will be unavoidable in case of the assumption common for Platonism and nominalism concerning single kind of existence. However, in mathematics neither nominalist negation of universals nor Platonic statement concerning existence independent of individuals are not accepted. The notions of unitarity and generality are associated in the determination of the class or species treated as predicates.[18] Therefore, if in factual sciences individualistic thesis can find justification, in mathematics the opposition between nominalism and realism concerning the conceptual object was overcome. However, this solution is not respected by the representatives of contempo-

[18] We shall say that for every predicate F, Fa iff $a \in \{x/Fx\}$.

rary nominalism (N. Goodman), rejecting all the versions of the set theory. In this way they put themselves outside the society of modern mathematicians.[19]

Contrary to Platonism, (mathematical) intuitionism (among others L. E. J. Brouwer, A. Heyting, H. Weyl) maintain that the objects of mathematics are constructs, products of human mind. They do not exist independent of cognizing subject. In the explanation of their nature and way of existence, one shall refer to appropriate cognitive acts and processes, effects of which are these objects. This ontological and, at the same time, psychological standpoint is characteristic for the authors, who do not accept objective idealism; it also penetrates naturalistic ontology (e.g. M. Bunge). It does not bind anybody to accept the epistemological thesis, according to which in each authentic concept some kind of intuition is contained. Certainly, the fact if certain concept or statement is intuitive is a clearly subjective problem, even considering the ambiguity of the term "intuition". For most of laymen, the statements like: a set of rational numbers is countable; there do exist non-commutative operations, e.g. matrix multiplication; will be incompatible with intuition. Therefore, the thesis concerning the absolute nature of intuitivity is empirically (psychologically) false. The qualification of exclusively intuitive constructs eliminates abstract creations. It was not without reason that Grassmann conception (Ausdehnungslehre, 1844), already containing important elements of abstract algebra and vector calculus was subjected to the criticism of contemporary representatives of Kantism by the reason of non-intuitivity and purely conceptual nature. The methodological thesis of intuitionism, according to which the only way to introduce mathematical constructs was their clear construction, may be expressed in two ways. On one hand, it is one of the basic strategies of theory construction, providing it does not exclude alternative strategies, and on the other, the thesis of constructivism is not philosophically justified as there is no ontological difference between real mathematical existence shown by the construction and pure mathematical existence generated by axiomatic definition or shown by reduction to absurd. These are two cases of conceptual or formal existence and not of physical one. On this account, a single number, set of numbers, or the power of such set are located on one plane, all of them are — in Platonic terminology — ideal objects, they are

[19] Presently, nominalism is represented only in philosophy and in Russian school of constructivism (e.g. A. Markov). On the issue of discussion with nominalism see: H. Putnam, *Philosophy of Logic*, New york 1971, p. 9 ff, and the review of this item written by Bas. C. van Fraassen, "Canadian Journal of Philosophy" 4(1975) pp. 731–743. See also J. Misiak, *Introduction* [in:] *Philosophy of Mathematics* (in Polish), Kraków 1986, p. 6.

physically irreal.

Mathematical empiricism, represented among others by E. Borel, F. Enriques, I. Lakatos, J. C. Harsanyi, P. Kitcher, possesses historical and philosophical versions. According to historical empiricism, every construct is generated by experience, however not directly, which does not leave space for pure mathematics. Within the scope of philosophical variant of mathematical empiricism[20] two theses are put forward. Every mathematical object is representing possible experience, as well as — possibly — the qualities of the surrounding world (semantic thesis). Mathematical research is carried out by the method of empirical research (therefore the method of trials and errors, induction, analogy are being used), and the final criterion for mathematical truth is constituted by experience (methodological thesis).[21]

In the discussion of the classical directions in the philosophy of mathematics above[22], the following items have been exposed in particular: (1) conceptual nature of objects and mathematical methods, without elimination of empirical or intuitive genesis of some of them; (2) universality of mathematical constructs created in sufficiently advanced societies of professionals; (3) differences between formal and factual statements, as well as between

[20] Empirist philosophy of geometry is presented by R. Torretti, *Philosophy of Geometry from Rieman to Poincaré*, Dordrecht 1978, p. 254 ff, on the example of its representatives (J. S. Mill, F. Überweg, R. Erdman, A. Calinon, E. Mach).

[21] M. Bunge, *Treatise*, pp. 111–119; A. Lubomirski, *On Generalization* (in Polish), pp. 44–45.

[22] Philosophy of mathematics is expected to solve a set of classical issues, which among others include the questions of the type: what is mathematics and what differentiates it from other sciences? what is the nature of mathematical objects and in what way do they differ from material objects? in what way do the mathematical objects exist? does mathematics have ontological assumptions? is it *a priori, a posteriori*, or does it contain both these elements? what is truth and proof in mathematics? what is its connection with logic? what is its attitude towards reality, accounting for variability of material objects and atemporal nature of mathematical objects?

Empirical issues undertaken by such factual disciplines like psychology and sociology of mathematics are out of the reach of mathematics. Within them the issues like e.g. the way of shaping and acquisition of mathematical ideas, the way of organizing itself, development and disintegration of the society of mathematicians are being discussed. The representatives of intuitionism, empiricism and dialectic materialism appear not to differentiate conceptual issues from empirical ones. It is an indispensable differentiation, although to understand mathematics as a whole requires conduction of both types of research. See: M. Bunge, *Treatise*, pp. 107–108; M. Lubański, *Z rozważań nad problemem prawdy w matematyce (From the Considerations upon the Issue of Truth in Mathematics)*, "Roczniki Filozoficzne" 32 (1984) fasc. 3 pp. 89, 101; R. Murawski, *"Humanizacja" w matematyce, tj. O nowych prądach w filozofii matematyki ("Humanization" of Mathematics, i.e. On New Currents in Philosophy of Mathematics)*, "Studia Filozoficzne" 8(249) 1986, p. 67 ff.

formal and empirical method of justification; (4) differences between the theory and its models, and between models in the semantic sense and the models, demonstrative in principle, occurring in science and technology; (5) formation of new constructs, establishment of connections and application of rules and algorithms without global treatment of mathematics as a tool of science and technology exclusively; (6) logical hierarchy of branches of mathematics (set theory, algebra, topology, etc.); (7) groundlessness of the conception of objects treated like independently existing Platonic ideas, as well as the conceptions of non-rational capabilities like intuition, providing we put aside its heuristic role.

Philosophy of mathematics is also considered here as an integral fragment of a philosophical system, comprehensive and compatible with science and technology. Creation in mathematics and its results are then perceptible in a few standard aspects. In particular, they are a finite product in the form of a set of theories (logical aspect). Semantic research is a specific type of mental activity with specific motivations and typical methods of reasoning (psychological aspect). They also constitute the manifestation of social activity, whose product is made by characteristic cultural artifacts (sociological aspect). Mathematics is a historical process, leading to discoveries propagated among society (historical aspect). Mathematical research constitutes also certain type of cognition, whose product is made by specific kind of knowledge (epistemological aspect).

These aspects of mathematics are in agreement, although they do not exclude one another. Therefore, it would be wrong to expose one of them excluding the rest. Mathematical research is carried out using all these alternative methods. Epistemological viewpoint seems to be prior to the rest, as it differentiates mathematics from among other types of scientific cognition. Fictionalism, being the equivalent of ontological fictionalism, has been formed on the basis of mathematical epistemology. The direction differs from the standpoint of F. Nietzsche and H. Vaihinger, as it is not justified on the ground of factual science and technology. Presently, Leibniz dualism of mental truths and factual truths is still being respected. Therefore, it is not the standpoint of philosophical monism, in which only one way of existence is justified.[23]

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[23] M. Bunge, *Treatise*, pp. 121–123; H. Putnam, *Philosophy of Logic* p. 63 ff; and review written by Bas C. van Frassen, p. 735. Bunge's fictionalism on the grounds of philosophy of mathematics is criticized by R. Toretti, *Three Kinds of Mathematical Fictionalism* [in:] *Scientific Philosophy Today*, Dordrecht 1982, pp. 399–414.